

# BPS states in type IIB SUGRA with $SO(4) \times SO(2)_{\text{gauged}}$ symmetry

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**ABSTRACT:** We present an extension of our construction (hep-th/0606199) exhibiting  $SO(4) \times SO(2)$  symmetry. We extend the previously presented ansatz by introducing a  $U(1)$  gauge field. The presence of the gauge field allows for more general values of the Killing spinor  $U(1)$  charge. One more time we identify a four dimensional Kahler structure and a Monge-Ampere type of equation parametrized by the  $U(1)$  Killing spinor charge. In addition we identify 2 scalars that parametrize the supersymmetric solutions, one of which is the Kahler potential.

**KEYWORDS:** AdS-CFT Correspondence, D-branes, Superstring Vacua.

JHEP05(2007)072

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## 1. Introduction

Supersymmetry provides us with a powerful tool in obtaining or at least describing solutions of supergravity theories. Using the G-structure analysis, originally developed in [1–5], one is able to write down the constraints that the bosonic fields need to satisfy in order for the background they create to be supersymmetric. In general the constraints are general and illuminating but the method is more fruitful when one makes a sensible reduction based on symmetry grounds. The method has been applied to several interesting configurations [6–11] and interesting results were obtained.

One of the most interesting cases was presented in [12] where the authors, among other results, demonstrated a one to one mapping between 1/2 BPS states in minimal type IIB supergravity and states in  $\mathcal{N} = 4$  SYM preserving the same amount of supersymmetry. The procedure exploits the fact that on both sides the states have the same moduli space which is parametrized by the phase space of  $N$  non-relativistic massless fermions in a simple harmonic potential. The field theory study was carried out in [13] and [14]. The symplectic form of the moduli space variables was later computed [15, 16] and was shown to agree with the symplectic form of the matrix model relevant to the field theory states [17].

In this paper we present an obvious extension of our previous effort [18] in generalizing the analysis of [12] to bosonic states preserving only  $SO(4) \times SO(2)$ . The new element that we are concerned with is the vector field that can be added to gauge the original  $SO(2)$ . Having a non-zero field strength allows us to have more general  $U(1)$  spinor charge, which is not constrained by the  $SO(4)$  chirality of the Killing spinor. As in the ungauged case we are able to demonstrate a four dimensional Kahler structure. In addition we are also able to show that the gauge vector can be parametrized by a single scalar function. We finally

show that the supersymmetry constraints give us a Monge-Ampere type of equation along with a non-linear constrain. The Bianchi identities that we want the five form to satisfy give us an additional constrain.

The paper is structured in three sections. In the first section we present our ansatz and we also show the technical highlights of the supersymmetry analysis. In the second section we embed known supersymmetric solutions in our general ansatz where the significance of the  $U(1)$  spinor charge becomes more transparent. In the last section we present a summary and conclusions. We also include an appendix where we give the technical details of the supersymmetry analysis for the interested reader.

## 2. The $SO(4) \times SO(2)$ symmetric ansatz and the SUSY analysis

In this section we will briefly describe the main steps of the supersymmetry analysis. Following the LLM analysis [12] we first reduce the ten dimensional theory by imposing  $SO(4) \times SO(2)$  symmetry on the fields of minimal type IIB supergravity, namely the metric  $g_{MN}$  and the self-dual five form field strength  $F_{M_1 M_2 M_3 M_4 M_5}$ . Our starting point is the ansatz

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu + e^{H+G} d\hat{\Omega}_3^2 + e^{H-G} (d\psi + A)^2 \\ F_{(5)} &= \hat{F}_{(2)} \wedge d\hat{\Omega}_3 + \tilde{F}_{(4)} \wedge (d\psi + A). \end{aligned} \tag{2.1}$$

Where the Greek indices  $\mu, \nu = 1 \dots 6$ .

In general, the constraints obtained by the G-structure analysis don't have to necessarily satisfy the field equations of type IIB supergravity. The check that guarantees the compatibility of the configuration with the type IIB field equations is the Bianchi identities that the five form should satisfy [19]. This argument is based on the integrability of the Killing spinor equation. For our case this means that the various form field strengths that come from the reduction of the five form have to satisfy

$$\hat{F}_2 = 2 e^{2G} e^H \star_6 \tilde{F}_4$$

and the Bianchi identity for the five form gives

$$\begin{aligned} d\hat{F}_{(2)} &= 0 \\ d\tilde{F}_{(4)} &= 0 \\ \tilde{F}_{(4)} \wedge \mathcal{F} &= 0. \end{aligned} \tag{2.2}$$

The problem that we would like to confront is to identify all the constraints imposed on the previously bosonic fields so that the Killing spinor equation

$$\mathcal{D}_{\mathcal{M}}\eta = \nabla_M \eta + \frac{i}{480} \Gamma^{M_1 \dots M_5} F_{M_1 \dots M_5} \Gamma_M \eta = 0. \tag{2.3}$$

will admit at least one non-trivial solution. After reducing on  $S^3 \times S^1$ , as we describe in appendix A we are left with a six dimensional spinor  $\varepsilon$ , a differential equation in six

dimensions and two algebraic ones coming from the reduction on  $S^3$  and  $S^1$  respectively

$$\left[ \nabla_\mu - \frac{in}{2} A_\mu + \frac{1}{4} e^{\frac{1}{2}(H-G)} \gamma_7 \gamma^\nu \mathcal{F}_{\nu\mu} - iN \gamma_\mu \right] \varepsilon = 0 \quad (2.4)$$

$$\left[ -\alpha e^{-\frac{1}{2}(H+G)} \gamma_7 + in e^{-\frac{1}{2}(H-G)} - \frac{1}{4} e^{\frac{1}{2}(H+G)} \not{\mathcal{F}} + \gamma_7 \gamma^\lambda \partial_\lambda H \right] \varepsilon = 0 \quad (2.5)$$

$$\left[ i\alpha e^{-\frac{1}{2}(H+G)} - n e^{-\frac{1}{2}(H-G)} \gamma_7 - i\gamma^\lambda \partial_\lambda G - \frac{i}{4} e^{\frac{1}{2}(H+G)} \not{\mathcal{F}} \gamma_7 + 4N \right] \varepsilon = 0 \quad (2.6)$$

$$N = -\frac{1}{4} \not{\hat{F}} e^{-\frac{3}{2}(G+H)} \quad (2.7)$$

where  $n$  is the  $U(1)$  spinor charge and  $\alpha$  is the  $SO(4)$  chirality.

At this point we introduce the spinor bilinears that one can construct from the six dimensional Killing spinor

$$f_1 = \bar{\varepsilon} \gamma_7 \varepsilon \quad (2.8)$$

$$f_2 = i \bar{\varepsilon} \varepsilon \quad (2.9)$$

$$K_\mu = \bar{\varepsilon} \gamma_\mu \varepsilon \quad (2.10)$$

$$L_\mu = \bar{\varepsilon} \gamma_\mu \gamma_7 \varepsilon \quad (2.11)$$

$$Y_{\mu\lambda} = i \bar{\varepsilon} \gamma_{\mu\nu} \gamma_7 \varepsilon \quad (2.12)$$

$$V_{\mu\nu} = \bar{\varepsilon} \gamma_{\mu\nu} \varepsilon \quad (2.13)$$

$$\Omega_{\mu\nu\lambda} = i \bar{\varepsilon} \gamma_{\mu\nu\lambda} \varepsilon. \quad (2.14)$$

As we show in appendix B one can prove that

$$\nabla_{(\mu} K_{\nu)} = 0$$

which suggests that  $K_\mu$  is a Killing vector for the six dimensional metric. At this point we impose the condition

$$K^\mu \mathcal{F}_{\mu\nu} = 0 \quad (2.15)$$

which will greatly simplify our analysis. One can then use the Killing spinor equation and the two projectors to show that

$$\begin{aligned} f_2 &= \kappa e^{\frac{1}{2}(H+G)} \\ f_1 &= \lambda e^{\frac{1}{2}(H-G)} \end{aligned}$$

where  $\kappa$  and  $\lambda$  are integration constants which give the same form for these bilinears similar to the ones we found in [18]. It is also notable that another consequence of (2.15) is the fact that  $L$  is again a closed form which we can show, by using (2.5), that

$$\partial_\mu e^H = -\frac{\alpha}{\lambda} L_\mu.$$

From the differential equation that  $L$  satisfies (B.15) we obtain the constrain

$$\mathcal{F}_{\lambda[\mu} V_{\nu]}^\lambda = 0$$

which will allow us to parametrize the field strength  $\mathcal{F}$  by a single scalar function. Using the six dimensional Fierz identities and the algebraic equation (2.5) can be used to show that the two vectors  $K$  and  $L$  are orthogonal and they have opposite norms

$$\begin{aligned} L^2 &= -K^2 = h^{-2} = f_1^2 + f_2^2 \\ L \cdot K &= 0 \end{aligned}$$

and also that

$$L^\mu \mathcal{F}_{\mu\nu} = 0.$$

Following similar arguments that were presented in similar configurations [8, 12, 18] one can reduce the form of the metric to

$$\begin{aligned} ds^2 &= -\frac{1}{h^2} (dt + C)^2 + h^2 dy^2 + e^{-G-H} h_{mn} dx^m dx^n + ye^G d\hat{\Omega}_3^2 + ye^{-G} d(\psi + A)^2, \\ &\quad m, n = 1, \dots, 4 \\ L &= -\alpha dy. \end{aligned}$$

At this point we have set  $\kappa = \lambda = 1$  and  $\gamma = -\alpha$ . Where the four dimensional metric  $h_{mn}$  also depends on the coordinate  $y$  as a parameter. The three form (2.14) which we show that can be written as

$$\Omega = e^{-G-H} K \wedge \mathcal{J}$$

provides us with a complex structure  $\mathcal{J}$  for the four dimensional base space that appears in the previous form of the metric. The  $y$  derivative of the Kahler form is given by a combination a the field strengths of the vectors  $C$  and  $A$

$$\frac{-\alpha}{y} \partial_y \mathcal{J} = \tilde{d}C - \mathcal{F}$$

while the  $y$  derivative of the vector  $C$  is given by

$$\partial_y C = -\frac{1}{y} \mathcal{J} \cdot \tilde{d}Z$$

where the exterior derivative  $\tilde{d}$  acts on the coordinates of the four dimensional space.

The four form can be shown that is given by

$$\begin{aligned} 4f_1 e^{-\frac{1}{2}(H-G)} \tilde{F}_{(4)} &= -\frac{2f_1^2}{f_1^2 + f_2^2} dG \wedge K \wedge I + f_1 e^{-\frac{1}{2}(G+H)} L \wedge K \wedge dC - (f_2^2 + f_1^2) dC \wedge I \\ &\quad + \lambda e^{H-G} I \wedge \mathcal{F} - e^{\frac{1}{2}(H-G)} \frac{f_1^2}{f_2} \frac{1}{f_1^2 + f_2^2} L \wedge K \wedge \mathcal{F} \end{aligned}$$

and the closure of this four form forces the constrain

$$\mathcal{F} \wedge d\left(e^{\frac{1}{2}(H-G)} Y\right) = 0$$

which can be shown to equivalent to the constrain (2.2).

The higher dimensional analog of the LLM Laplace equation is not altered in our case. As in the ungauged case, we find that the connection between the Kahler form and the scalar

$$Z = \frac{1}{2} \tanh G$$

is given by

$$y \partial_y \left( \frac{\partial_y \mathcal{J}}{y} \right) + \tilde{d} \left( \mathcal{J} \cdot \tilde{d} Z \right) = 0.$$

Supersymmetry demands that the volume of the four dimensional satisfies the differential equation

$$\partial_y \ln \det \mathcal{G} = 2 \frac{e^{-G}}{e^G + e^{-G}} \partial_y G + 2\alpha \frac{n + 2\alpha}{y(1 + e^{2G})} - \frac{2\alpha}{y} (n + \alpha)$$

which is a more general equation than what we had presented in [18]. Apart from the  $y$  derivative of the volume we would also like to consider the derivatives with respect to the coordinates of the four dimensional base space. For this reason we study the Ricci form of the four dimensional space. The quantity that will help us calculate the curvature of the Kahler manifold is the 2-form bilinear

$$\mathcal{V}_{\mu\nu} = \varepsilon^T \gamma_{\mu\nu} \gamma_7 \varepsilon.$$

This 2-form can be shown to be holomorphic and as in other interesting cases [6, 7] we can use the differential Killing spinor equation to calculate the curvature of the Kahler base space. In the end of the calculation we show that the Ricci form is given by (B.68).

The volume that correctly reproduces both equations under consideration is

$$\begin{aligned} \ln \left| \begin{array}{cc} \partial_z \partial_{\bar{z}} K & \partial_w \partial_{\bar{z}} K \\ \partial_z \partial_{\bar{w}} K & \partial_w \partial_{\bar{w}} K \end{array} \right| &= \ln \left( -y \partial_y \left( \frac{\partial_y K}{y} \right) + y F'(y) \right) + \frac{2\alpha}{y} (2\alpha + n) \partial_y K - 2\Phi \\ &\quad - 2\alpha (n + \alpha) \ln y - 2\alpha (2\alpha + n) F(y) \end{aligned} \tag{2.16}$$

where the function  $F(y)$  is such that

$$Z + \frac{1}{2} = -y \partial_y \left( \frac{\partial_y K}{y} \right) + y F'(y).$$

As we will see for the space-times which are asymptotically  $AdS_5 \times S^5$  this function is irrelevant. and the scalar  $\Phi$  is introduced through the field strength

$$\mathcal{F} = \tilde{d} \left( \mathcal{J} \cdot \tilde{d} \Phi \right)$$

which solves the constrain (B.17).

The scalar  $\Phi$  can be thought of as being sourced by the constrain

$$\mathcal{F} - \frac{1}{f_1^2 + f_2^2} i_L i_K \star_6 \mathcal{F} = -2e^{-2H} (n + \alpha) \mathcal{J} \tag{2.17}$$

which, generalizes the constrain  $n + \alpha = 0$  that appears in similar studies [8, 12, 18]. As we can see from the previous constrain, since both the field strength  $\mathcal{F}$  and the Kahler form

$\mathcal{J}$  are closed with respect to the four dimensional external differentiation operator  $\tilde{d}$ , we essentially have that

$$\tilde{d} \star_4 \mathcal{F} = 0$$

which is compatible with the type IIB field equations.

The two constraints that originate from the ten dimensional Bianchi identities are simultaneously equivalent to the constrain

$$\mathcal{F} \wedge \mathcal{F} = 2(n + 2\alpha)(n + \alpha) y^{-4} \mathcal{J} \wedge \mathcal{J} \quad (2.18)$$

which ensures that the geometry solves the ten dimensional type IIB equations of motion.

The last thing that needs to be checked is the closure of the two form  $\hat{F}_{(2)}$ . Using the duality relation (A.2) we can determine the two form  $\hat{F}_{(2)}$  whose components read

$$\begin{aligned} \hat{F}_{tx} &= -\frac{1}{2} e^{\frac{3}{2}(G+H)} f_2 \partial_x G \\ \hat{F}_{ty} &= -\frac{1}{4} \partial_y e^{2(G+H)} \\ \hat{F}_{yx} &= \frac{1}{4} C_x \partial_y e^{2(G+H)} - \frac{1}{8} e^{G+H} (e^G + e^{-G}) \mathcal{J}_x^{x_1} \partial_{x_1} z \\ \hat{F}_{x_1 x_2} &= \frac{1}{2} \mathcal{J}_{x_1 x_2} - \frac{1}{4} e^H \partial_y \mathcal{J}_{x_1 x_2} - \frac{1}{4} e^{2G+H} \partial_y \mathcal{J}_{x_1 x_2} - \frac{1}{4} e^{2(G+H)} \mathcal{F}_{x_1 x_2} - \frac{1}{4} \partial_{[x_1} e^{2(G+H)} C_{x_2]}. \end{aligned}$$

Checking the closure of the two form amounts to using the differential constraints that we have already identified.

To summarize the solution we write all the supergravity fields in terms of the two scalars  $\Phi$  and  $K$  in complex notation,

$$\begin{aligned} ds_{10}^2 &= -h^{-2} (dt + C)^2 + h^2 dy^2 + \frac{1}{ye^G} h_{mn} dx^m dx^n + ye^G d\hat{\Omega}_3^2 + ye^{-G} d(\psi + A)^2 \\ Z &= \frac{1}{2} \tanh G, \\ h^{-2} &= 2y \cosh G \\ Z &= -y \partial_y \left( \frac{\partial_y K}{y} \right), \\ h_{mn} dx^m dx^n &= \partial_{z^a} \partial_{\bar{z}^b} K dz^a d\bar{z}^b \\ C &= \frac{1}{2i} (\partial - \bar{\partial}) \left( \frac{1}{y} \partial_y K + \Phi \right), \\ A &= \frac{1}{2i} (\partial - \bar{\partial}) \Phi \end{aligned}$$

where  $\partial$  and  $\bar{\partial}$  are the holomorphic and antiholomorphic exterior derivatives respectively. keeping in mind that

$$\begin{aligned} K &= -h^{-2} (dt + C), \\ L &= dy \\ \mathcal{J} &= \partial \bar{\partial} K, \\ \mathcal{F} &= \partial \bar{\partial} \Phi \end{aligned}$$

the five form is

$$\begin{aligned}
 F_{(5)} &= 2 e^{2G} e^H \left( \star_6 \tilde{F}_{(4)} \right) \wedge d\hat{\Omega}_3 + \tilde{F}_{(4)} \wedge (d\psi + A) \\
 4\tilde{F}_{(4)} &= -h^2 e^{-2G} dG \wedge K \wedge \mathcal{J} + e^{-G} L \wedge K \wedge dC - h^{-2} e^{-G-H} dC \wedge \mathcal{J} \\
 &\quad + e^{-2G} \mathcal{J} \wedge \mathcal{F} - e^{H-2G} L \wedge K \wedge \mathcal{F}
 \end{aligned}$$

in terms of the two scalars the constraints (2.17) and (2.18) take the form

$$\begin{aligned}
 \partial\bar{\partial}\Phi - \star_4 \partial\bar{\partial}\Phi &= -\frac{2}{y^2} (n + \alpha) \partial\bar{\partial}K \\
 \partial\bar{\partial}\Phi \wedge \partial\bar{\partial}\Phi &= 2(n + 2\alpha)(n + \alpha) y^{-4} \partial\bar{\partial}K \wedge \partial\bar{\partial}K.
 \end{aligned}$$

### 3. Three examples

In this section we will reduce our general configuration to some known examples. This will help us illustrate the physical meaning of the  $U(1)$  charge  $n$  that enters in our analysis.

**Ungauged  $SO(2)$  ( $n = -\alpha$ ).** The first example that we present will reduce the more general ansatz (2.1) to the ungauged case we had originally considered in [18]. We take  $n = -\alpha$  and from the constrain (B.71) we see that the field strength of the gauge vector has to satisfy

$$\mathcal{F} \wedge \mathcal{F} = 0.$$

From our second constrain of the field strength (B.59) we also have that

$$\mathcal{F} = \star_4 \mathcal{F}$$

and because these forms carry only the four dimensional Euclidean indices we are forced to

$$\mathcal{F} = 0.$$

Finally, we see that the gauge field is a pure gauge which we can eliminate by shifting  $\psi$ . In this case we see that we recover the case of the ungauged solutions. As one can easily check the our more general equation (B.69) reduces to the equation we had proposed previously proposed in [18].

**LLM ( $n = -2\alpha$ ).** The class of LLM solutions [12] should be present in our previous case. However, writing down the correct Kahler potential and making the correct identification of coordinates looks like a rather non-trivial task. What we would like is to exploit is the presence of the gauge field.

$$\mathcal{F} \wedge \mathcal{F} = 0$$

but  $\mathcal{F}$  is non-zero now as we can see from the constraints that it satisfies. We split the four dimensional space in pairs  $(x_1, x_2)$  and  $(x_3, x_4)$ . We now write

$$\begin{aligned}
 \mathcal{F} &= A_3(x_3, x_4) dx^3 + A_4(x_3, x_4) dx^4 \\
 C &= C_1(x_1, x_2, y) dx^1 + C_2(x_1, x_2, y) dx^2 \\
 \mathcal{J} &= \mathcal{J}^{(1)} + y^2 \mathcal{J}^{(2)}
 \end{aligned}$$



and the metric takes the form

$$ds^2 = -\frac{1}{h^2} (dt + C)^2 + h^2 dy^2 + \frac{1}{ye^G} ds_1^2 + ye^G d\hat{\Omega}_3^2 + ye^{-G} \left[ ds_2^2 + \frac{1}{4} d(\tilde{\psi} + 2A)^2 \right].$$

Using equations (B.68) and (B.61) we have that for the two dimensional space spanned by  $(x_3, x_4)$  the Ricci tensor and the vector field  $A$  satisfy

$$\begin{aligned} R^{(2)} &= 4g^{(2)} \\ dA &= 2\mathcal{J}^{(2)}. \end{aligned}$$

From the first equation we see that the second two dimensional space is locally a two-sphere of radius  $\frac{1}{2}$  for which we may write

$$\mathcal{J}^{(2)} = \frac{1}{4} \sigma_1 \wedge \sigma_2$$

and from the second equation that we had for the vector field we see that

$$d(\tilde{\psi} + 2A) = d\sigma_3$$

where  $\sigma_i$  are the left invariant one forms on  $S^3$ . From equation (B.69) we have that

$$ds_1^2 = \left( Z + \frac{1}{2} \right) [dx_1^2 + dx_2^2]$$

and as we can see from (B.64) the function  $Z$  must satisfy

$$(\partial_1^2 + \partial_2^2) Z + y \partial_y \left( \frac{\partial_y Z}{y} \right) = 0.$$

Unlike the ungauged case  $n = -\alpha$  where even the very symmetric  $AdS_5 \times S^5$  looks very non-trivial, we see that after the inclusion of the  $U(1)$  fiber, the whole class of LLM solutions comes out very naturally.

**AdS<sub>5</sub> × Sasaki-Einstein** ( $n = -3\alpha$ ). In this case we have that

$$\mathcal{F} \wedge \mathcal{F} = 4y^{-4} \mathcal{J} \wedge \mathcal{J}$$

and we write the metric as

$$ds^2 = -\frac{1}{h^2} dt^2 + h^2 dy^2 + ye^G d\hat{\Omega}_3^2 + ye^{-G} \left[ ds_4^2(x^i) + d(\tilde{\psi} + A)^2 \right]$$

where we have rescaled pulled out a factor  $y^2$  from the four dimensional metric. In this case we set  $e^G = y$  and

$$Z = \frac{1}{2} \frac{1 - y^2}{1 + y^2}.$$

From the Ricci form of the four dimensional space (B.68) and equation (B.61) we can see that the four dimensional Kahler manifold becomes Einstein-Kahler

$$\mathcal{R} = 6\mathcal{J}.$$

On the other hand the Kahler form is related to the field strength  $\mathcal{F}$

$$\mathcal{F} = 2\mathcal{J}.$$

The above conditions guarantee that the 5 dimensional space spanned by the coordinates  $(\tilde{\psi}, x_1, \dots, x_4)$  is a five dimensional Sasaki-Einstein space. The final form of the metric is

$$ds^2 = -(y^2 + 1) dt^2 + \frac{dy^2}{y^2 + 1} + y^2 d\hat{\Omega}_3^2 + ds_4^2(x^i) + d(\tilde{\psi} + A)^2.$$

#### 4. Conclusions and summary

We have enlarged our previous  $SO(4) \times SO(2)$  symmetric ansatz [18] to include a vector field which gauges the  $SO(2)$  symmetry. The feature that persists after the inclusion of the new field is the four dimensional Kahler structure that we identified. This might not be very surprising since a six dimensional Kahler structure has been nicely identified in [6] for the  $SO(4)$  symmetric case. In addition to the Kahler potential a new scalar makes its appearance through the gauge field as someone would expect. The two scalars are coupled through constraints which makes them dependent.

The analysis we have presented in this paper is in no sense complete and more work is required. The configurations we have analyzed are all BPS but the constraints that one needs to solve are non-linear. It would be interesting to show at least the integrability of the above constraints and give a sense of superposition.

The question of gauge theory dual for the 1/4 and 1/8 [6] BPS configurations of supergravity is of great interest [20–25]. In the simplest case of 1/2 BPS, it was shown that the states were in 1-1 correspondence to gauge theory operators (through the fermion droplet picture). This picture was obtained through a reduction of gauge theory to 1-matrix model problem [13, 14]. For the present case one would need an analytical understanding of the large  $N$  limit of a multi-matrix model. It is known that for the case of 1/4 BPS states the model contains 2 complex matrices [26] while for 1/8 BPS states the model is described by 3 complex matrices. For a study of collective dynamics of such multi-matrix models see [21].

The configurations we have described are able to describe 1/4 BPS gravitons propagating on  $AdS_5 \times S^5$  which, at this point, we guess that are described by ripples on a four dimensional ball (for the  $n = -\alpha$  case) which should be interpreted as the  $AdS_5 \times S^5$  ground state. With our symmetry being  $SO(4) \times SO(2)$  we expect that, among other interesting setups, the configurations we analyzed describe the back reaction of rotating D3-branes wrapping the  $S^3$  of  $AdS_5$  with two independent angular momentum components on  $S^5$ . In this context it would be interesting to think of those geometries as a back reaction to Mikhailov's giant gravitons [27].

#### Acknowledgments

I would like to thank the MCTP high energy theory group for the hospitality during the beginning of this work. I would especially like to thank Sera Cremonini, Hai Lin, James Liu

and Diana Vaman for many useful discussions, comments and their collaboration. Finally, I thank Antal Jevicki and Robert McNees for many invaluable discussions.

## A. Killing spinor equation reduction

We start this appendix by giving the reduction of the spin connection on  $S^3 \times S^1$

$$\begin{aligned}\hat{\omega}_{cd} &= \omega_{cd} - \frac{1}{2}e^{H-G}\mathcal{F}_{cd}A - \frac{1}{2}e^{H-G}\mathcal{F}_{cd}d\psi \\ \hat{\omega}_{d\hat{m}} &= -\frac{1}{2}e^{\frac{1}{2}(H+G)}\hat{e}_{\hat{m}\hat{\mu}}\partial_d(H+G)dx^{\hat{\mu}} \\ \hat{\omega}_{10d} &= \frac{1}{2}e^{(H-G)}\mathcal{F}_{d\mu}dx^\mu + \frac{1}{2}e^{\frac{1}{2}(H-G)}\partial_d(H-G)A + \frac{1}{2}e^{\frac{1}{2}(H-G)}\partial_d(H-G)d\psi \\ \mathcal{F} &= dA\end{aligned}$$

where  $c, d = 1 \dots 6$  are the tangent indices of the dimensional space,  $\hat{m} = 1 \dots 3$  are the tangent indices of  $S^3$ , and  $\omega_{cd}$  is the spin connection of the six dimensional space.

We decompose the gamma matrices  $\Gamma_M$  in the following way

$$\begin{aligned}\Gamma_\mu &= \gamma_\mu \otimes \hat{\sigma}_1 \otimes \mathbb{I}_2 \\ \Gamma_{\hat{\mu}} &= \mathbb{I}_8 \otimes \hat{\sigma}_2 \otimes \sigma_{\hat{\mu}} \\ \Gamma_{10} &= \gamma_7 \otimes \hat{\sigma}_1 \otimes \mathbb{I}_2\end{aligned}$$

where  $\gamma_7 = \gamma_1 \dots \gamma_6$ ,  $\Gamma_{11} = \Gamma_1 \dots \Gamma_{10} = \mathbb{I}_8 \otimes \hat{\sigma}_3 \otimes \mathbb{I}_2$  and the chirality condition for the IIB spinors gives

$$\Gamma_{11}\eta = \eta \Rightarrow \hat{\sigma}_3\eta = \eta.$$

For the spinors we give the decomposition

$$\eta = \epsilon \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \chi_\alpha = \epsilon \otimes \chi_\alpha$$

where

$$\hat{\nabla}_{\hat{a}}\chi_\alpha = \frac{i\alpha}{2}\sigma_{\hat{a}}\chi_\alpha, \quad a = \pm 1.$$

For dependence on the coordinates we have

$$\epsilon(x^\mu, \hat{\Omega}_3, \psi) = e^{\frac{i}{2}n\psi}\epsilon(x^\mu, \hat{\Omega}_3)$$

which gives

$$i\partial_\psi\epsilon = -\frac{n}{2}\epsilon.$$

For the covariant derivatives we have

$$\begin{aligned}\hat{\nabla}_\mu &= \nabla_\mu - \frac{1}{8}e^{H-G}\not{\mathcal{F}}A_\mu + \frac{1}{4}e^{\frac{1}{2}(H-G)}\gamma_7\gamma^\nu\mathcal{F}_{\nu\mu} + \frac{1}{4}e^{\frac{1}{2}(H-G)}\gamma_7\gamma^\nu\partial_\nu(H-G)A_\mu \\ \nabla_{\hat{\mu}} &= \hat{\nabla}_{\hat{\mu}} - \frac{i}{4}\sigma_{\hat{\mu}}\hat{\sigma}_3\gamma^\nu\partial_\nu(G+H) \\ \nabla_\psi &= \partial_\psi - \frac{1}{8}e^{H-G}\not{\mathcal{F}} + \frac{1}{4}\gamma_7\gamma^\mu\partial_\mu(H-G)\end{aligned}$$

and for the Dirac matrices we have that

$$\Gamma_\mu = \gamma_\mu \hat{\sigma}_1 + e^{\frac{1}{2}(H-G)} A_\mu \gamma_7 \hat{\sigma}_1.$$

The ten dimensional Killing spinor equation reads

$$\nabla_M \eta + \frac{i}{480} \Gamma^{M_1 \dots M_5} F_{M_1 \dots M_5} \Gamma_M \eta = 0. \quad (\text{A.1})$$

For the second term we have

$$\begin{aligned} M &= \frac{i}{480} \Gamma^{M_1 \dots M_5} F_{M_1 \dots M_5} \\ &= \frac{i}{480} \left[ 10 \Gamma^{M_1 M_2} \hat{F}_{M_1 M_2} e^{-\frac{3}{2}(G+H)} \Gamma^{\hat{a} \hat{b} \hat{c}} \varepsilon_{\hat{a} \hat{b} \hat{c}} + 5 \Gamma^{M_1 M_2 M_3 M_4} \Gamma^\psi e^{-\frac{1}{2}(H-G)} \tilde{F}_{M_1 M_2 M_3 M_4} \right] \end{aligned}$$

after using the two duality relations

$$\Gamma^{M_1 \dots M_5} = -\frac{1}{6!} \varepsilon^{M_1 \dots M_{11}} \Gamma_{M_6 \dots M_{11}}$$

and

$$\tilde{F}_{M_1 \dots M_4} = -\frac{1}{2} e^{-2G-H} \varepsilon_{M_1 \dots M_6} \hat{F}^{M^5 M^6} \quad (\text{A.2})$$

we have

$$\begin{aligned} M &= -\frac{1}{8} \gamma^{\mu_1 \mu_2} \hat{F}_{\mu_1 \mu_2} e^{-\frac{3}{2}(G+H)} \hat{\sigma}_2 (1 + \Gamma_{11}) \\ &= -\frac{1}{8} \hat{F} \hat{\sigma}_2 (1 + \hat{\sigma}_3). \end{aligned}$$

The reduced equation now gives one differential and two algebraic constrains for the Killing spinor

$$\left[ \nabla_\mu - \frac{1}{8} e^{H-G} \not{F} A_\mu + \frac{1}{4} e^{\frac{1}{2}(H-G)} \gamma_7 \gamma^\nu W_{\nu\mu} \right. \quad (\text{A.3})$$

$$\left. + \frac{1}{4} e^{\frac{1}{2}(H-G)} \gamma_7 \gamma^\nu \partial_\nu (H-G) A_\mu - i N \gamma_\mu - i e^{\frac{1}{2}(H-G)} N \gamma_7 A_\mu \right] \varepsilon = 0$$

$$\left[ \frac{i\alpha}{2} e^{-\frac{1}{2}(H+G)} - \frac{i}{4} \gamma^\lambda \partial_\lambda (H+G) + N \right] \varepsilon = 0 \quad (\text{A.4})$$

$$\left[ \frac{m}{2} e^{-\frac{1}{2}(H-G)} - \frac{1}{8} e^{\frac{1}{2}(H-G)} \not{F} + \frac{1}{4} \gamma_7 \gamma^\lambda \partial_\lambda (H-G) - i \gamma_7 N \right] \varepsilon = 0 \quad (\text{A.5})$$

where

$$N = -\frac{1}{4} \hat{F} e^{-\frac{3}{2}(G+H)}.$$

The last equations can be written after linear combinations as

$$\begin{aligned} &\left[ \nabla_\mu - \frac{m}{2} A_\mu + \frac{1}{4} e^{\frac{1}{2}(H-G)} \gamma_7 \gamma^\nu \mathcal{F}_{\nu\mu} - i N \gamma_\mu \right] \varepsilon = 0 \\ &\left[ -\alpha e^{-\frac{1}{2}(H+G)} \gamma_7 + m e^{-\frac{1}{2}(H-G)} - \frac{1}{4} e^{\frac{1}{2}(H+G)} \not{F} + \gamma_7 \gamma^\lambda \partial_\lambda H \right] \varepsilon = 0 \\ &\left[ i \alpha e^{-\frac{1}{2}(H+G)} - n e^{-\frac{1}{2}(H-G)} \gamma_7 - i \gamma^\lambda \partial_\lambda G - \frac{i}{4} e^{\frac{1}{2}(H+G)} \not{F} \gamma_7 + 4N \right] \varepsilon = 0. \end{aligned}$$

## B. Supersymmetry analysis

When looking at Fierz identities involving the above bilinears it will be more convenient to work with the following equivalent bilinears

$$Z^+ = -f_1 - \iota f_2 = 2\varepsilon_+ \varepsilon_- \tag{B.1}$$

$$Z^- = f_1 - \iota f_2 = 2\varepsilon_- \varepsilon_+ \tag{B.2}$$

$$l_\mu^+ = L_\mu + K_\mu = 2\varepsilon_+ \gamma_\mu \varepsilon_+ \tag{B.3}$$

$$l_\mu^- = -L_\mu + K_\mu = 2\varepsilon_- \gamma_\mu \varepsilon_- \tag{B.4}$$

$$q_{\mu\nu\lambda}^\pm = \iota \varepsilon_\pm \gamma_{\mu\nu\lambda} \varepsilon_\pm \tag{B.5}$$

where as usually

$$\varepsilon_\pm = \frac{1}{2} (\mathbb{I}_8 \pm \gamma_7) \varepsilon \tag{B.6}$$

$$\gamma_7 \varepsilon_\pm = \pm \varepsilon_\pm. \tag{B.7}$$

We will also make extensive use of the duality relation in six dimensions between the gamma matrices

$$\gamma^{a_1 \dots a_n} = \frac{(-1)^{\lfloor \frac{n}{2} \rfloor + 1}}{(6-n)!} \varepsilon^{a_1 \dots a_n b_{6+1} \dots b_{6-n}} \gamma_{b_1 \dots b_{6-n}} \gamma_7 \tag{B.8}$$

and the Fierz rearrangement for commuting spinors

$$\begin{aligned} \bar{\psi}_1 \psi_2 \bar{\psi}_3 \psi_4 &= \frac{1}{8} \left[ \bar{\psi}_1 \psi_4 \bar{\psi}_3 \psi_2 + \bar{\psi}_1 \gamma_7 \psi_4 \bar{\psi}_3 \gamma_7 \psi_2 - \frac{1}{2} \bar{\psi}_1 \gamma_{\mu\nu} \psi_4 \bar{\psi}_3 \gamma^{\mu\nu} \psi_2 - \frac{1}{2} \bar{\psi}_1 \gamma_\mu \gamma_7 \psi_4 \bar{\psi}_3 \gamma^{\mu\nu} \gamma_7 \psi_2 \right] \\ &+ \frac{1}{8} \left[ \bar{\psi}_1 \gamma_\mu \psi_4 \bar{\psi}_3 \gamma^\mu \psi_2 - \bar{\psi}_1 \gamma_\mu \gamma_7 \psi_4 \bar{\psi}_3 \gamma^\mu \gamma_7 \psi_2 \right] \\ &- \frac{1}{96} \left[ \bar{\psi}_1 \gamma_{\mu\nu\lambda} \psi_4 \bar{\psi}_3 \gamma^{\mu\nu\lambda} \psi_2 - \bar{\psi}_1 \gamma_{\mu\nu\lambda} \gamma_7 \psi_4 \bar{\psi}_3 \gamma^{\mu\nu\lambda} \gamma_7 \psi_2 \right]. \end{aligned} \tag{B.9}$$

### B.1 Diferential relations

Taking a covariant derivative of (2.8) and using equation (2.4) we may write

$$\begin{aligned} \nabla_\mu f_1 &= \frac{\iota}{4} \bar{\varepsilon} (\gamma_\mu \gamma_{\kappa\lambda} \gamma_7 + \gamma_{\kappa\lambda} \gamma_\mu \gamma_7) \varepsilon F^{\kappa\lambda} e^{-\frac{3}{2}(G+H)} - \frac{1}{2} e^{\frac{1}{2}(H-G)} \bar{\varepsilon} \gamma^\nu \varepsilon \mathcal{F}_{\nu\mu} \\ &= \frac{1}{2} \star (F \wedge \Omega)_\mu e^{-\frac{3}{2}(G+H)} - \frac{1}{2} e^{\frac{1}{2}(H-G)} K^\nu \mathcal{F}_{\nu\mu} \end{aligned} \tag{B.10}$$

$$= -\frac{1}{3!} F_{\mu\rho\sigma\tau} \Omega^{\rho\sigma\tau} e^{\frac{1}{2}(G-H)} - \frac{1}{2} e^{\frac{1}{2}(H-G)} K^\nu \mathcal{F}_{\nu\mu} \tag{B.11}$$

Following the same procedure for (2.9) we have

$$\begin{aligned} \nabla_\mu f_2 &= -\frac{1}{2} \bar{\varepsilon} (\gamma_\lambda g_{\mu\kappa} - g_{\mu\lambda} \gamma_\kappa) \varepsilon F^{\kappa\lambda} e^{-\frac{3}{2}(G+H)} \\ &= -e^{-\frac{3}{2}(G+H)} F_{\mu\lambda} K^\lambda. \end{aligned} \tag{B.12}$$

For (2.10) we have

$$\nabla_\mu K_\rho = e^{-\frac{3}{2}(G+H)} f_2 F_{\mu\rho} - e^{-\frac{3}{2}(G+H)} \frac{1}{4} F^{\kappa\lambda} \varepsilon_{\mu\rho\pi\tau\kappa\lambda} Y^{\pi\tau} + \frac{1}{2} e^{\frac{1}{2}(H-G)} f_1 \mathcal{F}_{\lambda\mu} \quad (\text{B.13})$$

$$= e^{-\frac{3}{2}(G+H)} f_2 F_{\mu\rho} + \frac{1}{2} e^{\frac{1}{2}(G-H)} F_{\mu\rho\pi\tau} Y^{\pi\tau} + \frac{1}{2} e^{\frac{1}{2}(H-G)} f_1 \mathcal{F}_{\lambda\mu} \quad (\text{B.14})$$

For (2.11) we have

$$\begin{aligned} \nabla_\mu L_\rho &= \frac{i}{4} F^{\kappa\lambda} e^{-\frac{3}{2}(G+H)} \bar{\varepsilon} (\gamma_\rho \gamma_{\kappa\lambda} \gamma_\mu + \gamma_\mu \gamma_{\kappa\lambda} \gamma_\rho) \gamma_7 \varepsilon + \frac{1}{4} e^{\frac{1}{2}(H-G)} \bar{\varepsilon} (\gamma^\nu \gamma_\rho - \gamma_\rho \gamma^\nu) \varepsilon \mathcal{F}_{\nu\mu} \\ &= e^{-\frac{3}{2}(G+H)} \left[ F_\mu^\lambda Y_{\lambda\rho} + F_\rho^\lambda Y_{\lambda\mu} + \frac{1}{2} g_{\mu\rho} F^{\kappa\lambda} Y_{\kappa\lambda} \right] - \frac{1}{2} e^{\frac{1}{2}(H-G)} V_\rho^\nu \mathcal{F}_{\nu\mu} \end{aligned} \quad (\text{B.15})$$

where we used the duality (B.8). One more equation for (2.13) is given by

$$\nabla_\gamma V_{\delta\varepsilon} = -e^{-\frac{3}{2}(G+H)} \left[ g_{\gamma[\varepsilon} \Omega_{\delta]\alpha\beta} F^{\alpha\beta} - F_\gamma^\beta \Omega_{\beta\delta\varepsilon} + 2\Omega_{\alpha\gamma[\delta} F_{\varepsilon]}^\alpha \right] + e^{\frac{1}{2}(H-G)} L_\gamma \mathcal{F}_{\delta\varepsilon}. \quad (\text{B.16})$$

From equation (B.13) we conclude that

$$\nabla_{(\mu} K_{\nu)} = 0$$

which suggests that  $K_\mu$  is a Killing vector.

As we will later see in (B.35), a consequence of (2.15), the 1-form  $L_\mu$  is the derivative of a scalar and so we have from (B.15) the constraint

$$dL = 0 \Rightarrow \mathcal{F}_{\lambda[\mu} V_{\nu]}^\lambda = 0. \quad (\text{B.17})$$

We now take a derivative of (2.14) giving us

$$\nabla_\kappa \Omega_{\mu\nu\lambda} = \frac{1}{4} e^{-\frac{3}{2}(G+H)} F^{\pi\rho} \bar{\varepsilon} (\gamma_{\mu\nu\lambda} \gamma_{\pi\rho} \gamma_\kappa - \gamma_\kappa \gamma_{\pi\rho} \gamma_{\mu\nu\lambda}) \varepsilon + \frac{3}{2} e^{\frac{1}{2}(H-G)} Y_{[\mu\nu} \mathcal{F}_{\lambda]\kappa}.$$

After antisymmetrization we have that

$$d\Omega_{\kappa\lambda\mu\nu} = 4f_1 e^{-\frac{1}{2}(H-G)} \tilde{F}_{\kappa\lambda\mu\nu} + 6 e^{\frac{1}{2}(H-G)} Y_{[\mu\nu} \mathcal{F}_{\lambda\kappa]}. \quad (\text{B.18})$$

For its dual we have the equation

$$\nabla_\kappa (\star\Omega)_{\mu\nu\lambda} = -\frac{1}{4} e^{-\frac{3}{2}(G+H)} F^{\pi\rho} \bar{\varepsilon} (\gamma_{\mu\nu\lambda} \gamma_{\pi\rho} \gamma_\kappa + \gamma_\kappa \gamma_{\pi\rho} \gamma_{\mu\nu\lambda}) \gamma_7 \varepsilon - \frac{i}{2} e^{\frac{1}{2}(H-G)} \bar{\varepsilon} \gamma_{\mu\nu\lambda}{}^\rho \varepsilon \mathcal{F}_{\rho\kappa}.$$

Antisymmetrizing the last equation in  $\kappa, \mu, \nu, \lambda$  we obtain

$$(d\star\Omega)_{\kappa\mu\nu\lambda} = 2\bar{\varepsilon} (\gamma_{\mu\nu\lambda\kappa} N + N \gamma_{\kappa\mu\nu\lambda}) \gamma_7 \varepsilon - e^{\frac{1}{2}(H-G)} \mathcal{F}_{[\kappa}^\pi \varepsilon_{\mu\nu\lambda]\pi\alpha\beta} Y^{\alpha\beta} \quad (\text{B.19})$$

which as we see later promotes a 4-dimensional submanifold to a Kahler manifold.

**Algebraic relations.** Using  $\bar{\psi}_1 = \bar{\varepsilon}_+ \gamma_\mu$ ,  $\psi_2 = \varepsilon_+$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_+$  and  $\psi_4 = \gamma_\mu \varepsilon_+$  in (B.9) we obtain

$$l_\mu^+ l^{+\mu} = 0. \quad (\text{B.20})$$

In a similar manner one can obtain the relation

$$l_\mu^- l^{-\mu} = 0. \quad (\text{B.21})$$

Using  $\bar{\psi}_1 = \bar{\varepsilon}_+$ ,  $\psi_2 = \varepsilon_-$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_-$  and  $\psi_4 = \varepsilon_+$  we obtain

$$Z^+ Z^- = \frac{1}{4} l_\mu^+ l^{-\mu} + \frac{1}{48} q^{+\pi\rho\sigma} q_{\pi\rho\sigma}^-. \quad (\text{B.22})$$

Choosing  $\bar{\psi}_1 = \bar{\varepsilon}_+ \gamma^\mu$ ,  $\psi_2 = \varepsilon_+$ ,  $\bar{\psi}_3 = i\bar{\varepsilon}_+$  and  $\psi_4 = \gamma_{\mu\nu\lambda} \varepsilon_+$  we have

$$l^{+\mu} q_{\mu\nu\lambda}^+ = -l^{+\mu} q_{\mu\nu\lambda}^+ \Rightarrow i_{l^+} q^+ = 0. \quad (\text{B.23})$$

Using the selfduality

$$\star q^+ = q^+$$

we have also that

$$l^+ \wedge q^+ = 0.$$

In a very similar way one can prove that

$$i_{l^-} q^- = 0$$

and from the anti-selfduality

$$\star q^- = -q^-$$

follows that

$$l^- \wedge q^- = 0.$$

The above two relations lead us to the conclusion that in a frame where the metric can be written as

$$ds^2 = e^+ e^- + \delta_{ab} e^a e^b, \quad a, b = 1 \dots 4 \quad (\text{B.24})$$

we must have

$$l^+ = e^+$$

$$l^- = e^-$$

and for the three forms

$$q^+ = l^+ \wedge I \quad (\text{B.25})$$

$$q^- = l^- \wedge J \quad (\text{B.26})$$

where

$$I = \frac{1}{2} I_{ab} e^a \wedge e^b \quad (\text{B.27})$$

$$J = \frac{1}{2} J_{ab} e^a \wedge e^b. \quad (\text{B.28})$$

From the dualities of  $q^\pm$  we conclude that  $I$  and  $J$  are anti-selfdual with respect to the four dimensional space. From (B.25) and (B.26) we conclude that

$$\begin{aligned} q^{+\pi\rho\sigma} q_{\pi\rho\sigma}^- &= 3 \left[ (l^+ \cdot l^-) \left( I^{ab} J_{ab} \right) + 2l^{+m} l^{-n} I_{nk} J_m^k \right]. \\ &= 3 (l^+ \cdot l^-) \left( I^{ab} J_{ab} \right). \end{aligned} \quad (\text{B.29})$$

We are now considering (A.4), (A.5) and their conjugates

$$\begin{aligned} \left[ \frac{\imath\alpha}{2} e^{-\frac{1}{2}(H+G)} - \frac{\imath}{4} \gamma^\lambda \partial_\lambda (H+G) + N \right] \varepsilon &= 0 \\ \bar{\varepsilon} \left[ -\frac{\imath\alpha}{2} e^{-\frac{1}{2}(H+G)} - \frac{\imath}{4} \gamma^\lambda \partial_\lambda (G+H) - N \right] &= 0 \end{aligned} \quad (\text{B.30})$$

$$\begin{aligned} \left[ \frac{\imath n}{2} e^{-\frac{1}{2}(H-G)} + \frac{1}{4} \gamma_\tau \gamma^\lambda \partial_\lambda (H-G) - \frac{1}{8} e^{\frac{1}{2}(H-G)} \not{F} - \imath \gamma_\tau N \right] \varepsilon &= 0 \\ \bar{\varepsilon} \left[ -\frac{\imath n}{2} e^{-\frac{1}{2}(H-G)} + \frac{1}{4} \gamma^\lambda \gamma_\tau \partial_\lambda (H-G) + \frac{1}{8} e^{\frac{1}{2}(H-G)} \not{F} + \imath \gamma_\tau N \right] &= 0. \end{aligned} \quad (\text{B.31})$$

**Vector identities.** At this point it is useful to see that

$$[\gamma_\mu, N] = -\frac{1}{2} e^{-\frac{3}{2}(G+H)} F_\mu^\lambda \gamma_\lambda.$$

Multiplying (A.4) by  $\bar{\varepsilon} \gamma_\mu$ , (B.30) by  $\gamma_\mu \varepsilon$  and adding the two equations we have

$$\begin{aligned} \bar{\varepsilon} \left[ \frac{\imath}{4} \left\{ \gamma_\mu, \gamma^\lambda \right\} \partial_\lambda (H+G) + [\gamma_\mu, N] \right] \varepsilon &= 0 \Rightarrow \\ f_2 \partial_\mu (H+G) - e^{-\frac{3}{2}(G+H)} F_\mu^\lambda K_\lambda &= 0 \end{aligned} \quad (\text{B.32})$$

which in combination with (B.12) gives the equation

$$\begin{aligned} \partial_\mu f_2 &= \frac{1}{2} f_2 \partial_\mu (H+G) \Rightarrow \\ f_2 &= \kappa e^{\frac{1}{2}(H+G)}. \end{aligned} \quad (\text{B.33})$$

We now turn to (A.5) and (B.31), we multiply the first by  $\bar{\varepsilon} \gamma_\mu$ , the second one by  $\gamma_\mu \varepsilon$  we add them and in combination with (2.15) yields

$$\begin{aligned} \bar{\varepsilon} \left[ -\frac{1}{4} \gamma_\tau \left\{ \gamma_\mu, \gamma^\lambda \right\} \partial_\lambda (H-G) - \imath \left\{ \gamma_\mu, N \right\} \gamma_\tau \right] \varepsilon &= 0 \\ -f_1 \partial_\mu (H-G) + \star (F \wedge \Omega)_\mu e^{-\frac{3}{2}(G+H)} &= 0 \end{aligned}$$

which in combination with (B.10) yields

$$\begin{aligned} \partial_\mu f_1 &= \frac{1}{2} f_1 \partial_\mu (H-G) \Rightarrow \\ f_1 &= \lambda e^{\frac{1}{2}(H-G)}. \end{aligned} \quad (\text{B.34})$$

We now consider (A.4) and (B.30) again but this time we multiply by  $-\imath \bar{\varepsilon} \gamma_\mu \gamma_\tau$  and  $-\imath \gamma_\mu \gamma_\tau \varepsilon$  respectively and add them. The result of the operation reads

$$\alpha L_\mu e^{-\frac{1}{2}(G+H)} + \frac{1}{2} f_1 \partial_\mu (G+H) + \frac{1}{2} \star (\Omega \wedge F)_\mu e^{-\frac{3}{2}(G+H)} = 0.$$



Using (B.10) and B.34 we obtain the relation

$$\partial_\mu e^H = -\frac{\alpha}{\lambda} L_\mu. \quad (\text{B.35})$$

Multiplying A.5 and B.31 by  $i\bar{\varepsilon}\gamma_\mu\gamma_7$  respectively  $i\gamma_\mu\gamma_7\varepsilon$  and adding the resulting equations we obtain

$$-ne^{-\frac{1}{2}(H-G)}L_\mu - \frac{1}{4}e^{\frac{1}{2}(H-G)}\star\Omega_{\mu\lambda\nu}\mathcal{F}^{\lambda\nu} + \frac{1}{2}f_2\partial_\mu(H-G) - \frac{1}{2}e^{-\frac{3}{2}(G+H)}F_\mu^\lambda K_\lambda = 0.$$

We now use B.12 and B.33 to obtain

$$\partial_\mu e^H = \frac{n}{\kappa}L_\mu + \frac{1}{4}e^{\frac{1}{2}(H-G)}\star\Omega_{\mu\lambda\nu}\mathcal{F}^{\lambda\nu} \quad (\text{B.36})$$

which in combination with (B.35) gives the constraint

$$\star\Omega_{\mu\lambda\nu}\mathcal{F}^{\lambda\nu} = -4e^{-H+G}\left(\frac{n}{\kappa} + \frac{\alpha}{\lambda}\right)L_\mu.$$

From equation (2.5) one can also derive the constraints

$$\begin{aligned} -\alpha e^{-\frac{1}{2}(G+H)}K_\mu + V_\mu^\lambda\partial_\lambda H - \frac{1}{2}e^{\frac{1}{2}(H-G)}\mathcal{F}_{\mu\lambda}L^\lambda &= 0 \\ ne^{-\frac{1}{2}(H-G)}K_\mu + Y_\mu^\lambda\partial_\lambda H + \frac{1}{4}e^{\frac{1}{2}(H-G)}\Omega_{\mu\lambda\nu}\mathcal{F}^{\lambda\nu} &= 0 \end{aligned}$$

**Rank three identities.** We now multiply (2.5) by  $\bar{\varepsilon}\gamma_{\mu\nu\kappa}$ ,  $\bar{\varepsilon}\gamma_{\mu\nu\kappa}\gamma_7$  and take the real and imaginary part separately

$$\begin{aligned} \alpha e^{-\frac{1}{2}(G+H)}\Omega_{\mu\nu\kappa} + \frac{1}{2}\epsilon_{\mu\nu\kappa\rho\alpha\beta}Y^{\alpha\beta}\partial^\rho H + \frac{3}{2}e^{\frac{1}{2}(H-G)}\mathcal{F}_{[\kappa}{}^\beta\star\Omega_{\mu\nu]\beta} &= 0 \\ ne^{-\frac{1}{2}(H-G)}\star\Omega_{\mu\nu\kappa} + 3\partial_{[\kappa}HV_{\mu\nu]} + \frac{3}{2}e^{\frac{1}{2}(H-G)}L_{[\kappa}\mathcal{F}_{\mu\nu]} + \frac{1}{4}e^{\frac{1}{2}(H-G)}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}K^\gamma\mathcal{F}^{\alpha\beta} &= 0 \end{aligned} \quad (\text{B.37})$$

$$\alpha e^{-\frac{1}{2}(G+H)}\star\Omega_{\mu\nu\kappa} + 3\partial_{[\kappa}HY_{\mu\nu]} + \frac{3}{2}e^{\frac{1}{2}(H-G)}\mathcal{F}_{[\kappa}{}^\beta\Omega_{\mu\nu]\beta} = 0 \quad (\text{B.38})$$

$$ne^{-\frac{1}{2}(H-G)}\Omega_{\mu\nu\kappa} + \frac{1}{2}\epsilon_{\mu\nu\kappa\rho\alpha\beta}V^{\alpha\beta}\partial^\rho H + \frac{3}{2}e^{\frac{1}{2}(G+H)}K_{[\kappa}\mathcal{F}_{\mu\nu]} + \frac{1}{4}e^{\frac{1}{2}(H-G)}\epsilon_{\kappa\mu\nu\alpha\beta\gamma}L^\gamma\mathcal{F}^{\alpha\beta} = 0 \quad (\text{B.39})$$

**Kahler Structure.** At this point we would like to recognize a Kahler structure in the construction at hand. For this reason we consider (B.9) with  $\bar{\psi}_1 = \bar{\varepsilon}_\pm\gamma^{\mu\nu}\gamma^\alpha$ ,  $\psi_2 = \varepsilon_\pm$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_\pm$  and  $\psi_4 = \gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm$  which gives

$$\begin{aligned} 4\bar{\varepsilon}_\pm\gamma^{\mu\nu}\gamma^\alpha\varepsilon_\pm\bar{\varepsilon}_\pm\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm &= \bar{\varepsilon}_\pm\gamma^{\mu\nu}\gamma^\alpha\gamma_\rho\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm\bar{\varepsilon}_\pm\gamma^\rho\varepsilon_\pm - \frac{1}{12}\bar{\varepsilon}_\pm\gamma^{\mu\nu}\gamma^\alpha\gamma_{\rho\sigma\tau}\gamma_\alpha\gamma^{\gamma\delta}\varepsilon_\pm\bar{\varepsilon}_\pm\gamma^{\rho\sigma\tau}\varepsilon_\pm \\ &= -4\bar{\varepsilon}_\pm\gamma^{\mu\nu}\gamma_\rho\gamma^{\gamma\delta}\varepsilon_\pm\bar{\varepsilon}_\pm\gamma^\rho\varepsilon_\pm. \end{aligned}$$

Using the identities

$$\gamma_{\mu\nu}\gamma_\alpha = g_{\alpha\nu}\gamma_\mu - g_{\alpha\mu}\gamma_\nu + \gamma_{\alpha\mu\nu} \quad (\text{B.40})$$

$$\gamma_\alpha\gamma_\gamma\delta = g_{\alpha\gamma}\gamma_\delta - g_{\alpha\delta}\gamma_\gamma + \gamma_{\alpha\gamma\delta} \quad (\text{B.41})$$

$$\begin{aligned} \gamma_{\mu\nu}\gamma_\alpha\gamma_\gamma\delta &= \gamma_{\alpha\gamma\delta\mu\nu} - 6g_{\alpha\pi}g_{\gamma\rho}g_{\delta\sigma}\delta_{\mu\nu}^{[\pi\rho}\gamma^{\sigma]} + 4g_{\alpha\pi}g_{\mu\rho}g_{\nu\sigma}\delta_{\gamma\delta}^{\pi[\rho}\gamma^{\sigma]} \\ &\quad + 6g_{\alpha\pi}g_{\gamma\rho}g_{\delta\sigma}\delta_{[\nu}^{\pi}\gamma^{\rho\sigma]}_{\mu]} + 2g_{\alpha[\gamma}\gamma_{\delta]\mu\nu} \end{aligned}$$

$$\gamma^\alpha\gamma_{\rho\sigma\tau}\gamma_\alpha = 0$$

we draw the conclusion

$$I_b^a I_c^b = -\delta_c^a \quad (\text{B.42})$$

$$J_b^a J_c^b = -\delta_c^a.$$

Using (B.20), (B.21), (B.42), (B.22), (B.29) and the definitions (B.1)-(B.4) we have that

$$L^2 = -K^2 = f_1^2 + f_2^2 \quad (\text{B.43})$$

$$K \cdot L = 0. \quad (\text{B.44})$$

We now look at the Fierz identity involving  $\bar{\psi}_1 = \bar{\varepsilon}_+\gamma^{\gamma\delta}\gamma_\mu$ ,  $\psi_2 = \varepsilon_+$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_-$  and  $\psi_4 = \gamma_\nu\gamma_\gamma\delta\varepsilon_-$  which gives after antisymmetrization in  $\mu$  and  $\nu$

$$\begin{aligned} 4\bar{\varepsilon}_+\gamma^{\gamma\delta}_{[\mu}\varepsilon_+\bar{\varepsilon}_-\gamma_{\nu]}\gamma_\delta\varepsilon_- &= -12\bar{\varepsilon}_+\gamma_{\mu\nu}\varepsilon_-\bar{\varepsilon}_-\varepsilon_+ + 12\bar{\varepsilon}_+\varepsilon_-\bar{\varepsilon}_-\gamma_{\mu\nu}\varepsilon_+ \\ &\quad - 2\bar{\varepsilon}_+\gamma_{\mu\nu\rho\sigma}\varepsilon_-\bar{\varepsilon}_-\gamma^{\rho\sigma}\varepsilon_+, \end{aligned} \quad (\text{B.45})$$

considering now the Fierz identity for  $\bar{\psi}_1 = \bar{\varepsilon}_+\gamma_\nu$ ,  $\psi_2 = \varepsilon_+$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_-$  and  $\psi_4 = \gamma_\mu\varepsilon_-$  we have

$$\bar{\varepsilon}_+\gamma_{\mu\nu\rho\sigma}\varepsilon_-\bar{\varepsilon}_-\gamma^{\rho\sigma}\varepsilon_+ = 8\bar{\varepsilon}_+\gamma_{[\nu}\varepsilon_+\bar{\varepsilon}_-\gamma_{\mu]}\varepsilon_- + 2\bar{\varepsilon}_+\gamma_{\mu\nu}\varepsilon_-\bar{\varepsilon}_-\varepsilon_+ - 2\bar{\varepsilon}_+\varepsilon_-\bar{\varepsilon}_-\gamma_{\mu\nu}\varepsilon_+.$$

Finally (B.45) takes the form

$$\bar{\varepsilon}_+\gamma^{\gamma\delta}_{[\mu}\varepsilon_+\bar{\varepsilon}_-\gamma_{\nu]}\gamma_\delta\varepsilon_- = -4\bar{\varepsilon}_+\gamma_{\mu\nu}\varepsilon_-\bar{\varepsilon}_-\varepsilon_+ + 4\bar{\varepsilon}_+\varepsilon_-\bar{\varepsilon}_-\gamma_{\mu\nu}\varepsilon_+ - 4\bar{\varepsilon}_+\gamma_{[\nu}\varepsilon_+\bar{\varepsilon}_-\gamma_{\mu]}\varepsilon_-. \quad (\text{B.46})$$

It will be also useful to consider the Fierz identities with the choice  $\bar{\psi}_1 = \bar{\varepsilon}_-\gamma^\mu$ ,  $\psi_2 = \varepsilon_-$ ,  $\bar{\psi}_3 = \bar{\varepsilon}_-$  and  $\psi_4 = \gamma_\mu\gamma_\nu\varepsilon_+$  which gives the relation

$$\begin{aligned} \bar{\varepsilon}_-\gamma^\mu\varepsilon_-\bar{\varepsilon}_-\gamma_\mu\gamma_\nu\varepsilon_+ &= 0 \Rightarrow \\ \bar{\varepsilon}_-\gamma^\mu\varepsilon_-\bar{\varepsilon}_-\gamma_{\mu\nu}\varepsilon_+ &= -\bar{\varepsilon}_-\gamma_\nu\varepsilon_-\bar{\varepsilon}_-\varepsilon_+. \end{aligned} \quad (\text{B.47})$$

In a similar way one can derive

$$\bar{\varepsilon}_+\gamma^\mu\varepsilon_+\bar{\varepsilon}_+\gamma_{\mu\nu}\varepsilon_- = -\bar{\varepsilon}_+\gamma_\nu\varepsilon_+\bar{\varepsilon}_+\varepsilon_-. \quad (\text{B.48})$$

Contracting (B.39) with  $K^\mu L^\nu$  and using (B.35), (B.44) and (2.15) we obtain

$$L^\mu \mathcal{F}_{\mu\nu} = 0. \quad (\text{B.49})$$

Contracting (B.39) with  $L^\nu$  we obtain the equation

$$n\Omega_{\mu\lambda\nu}L^\nu = 0$$

which after combining with (B.36) gives

$$n(i_{l^+}q^- - i_{l^-}q^+) = 0 \tag{B.50}$$

where we used

$$i_{l^\pm}q^\pm = 0.$$

From (B.50), (B.25) and (B.26) we conclude that

$$n(I - J) = 0.$$

From now on we assume that  $n \neq 0$  and we conclude that

$$I = J.$$

Using equations (A.4), (A.5), (B.30) and (B.31) one can show that

$$\bar{\varepsilon}(\gamma_{\mu\nu\lambda\kappa}N + N\gamma_{\kappa\mu\nu\lambda})\gamma\tau\varepsilon = \frac{1}{2}(\star\Omega \wedge dG)_{\kappa\mu\nu\lambda} - \frac{1}{2}e^{\frac{1}{2}(H-G)}\mathcal{F}^\pi_{[\kappa\varepsilon_{\mu\nu\lambda]}\pi\alpha\beta}Y^{\alpha\beta}.$$

Combining the above equation with (B.19) we have that

$$d\star\Omega = \star\Omega \wedge dG - 2e^{\frac{1}{2}(H-G)}\mathcal{F}^\pi_{[\kappa\varepsilon_{\mu\nu\lambda]}\pi\alpha\beta}Y^{\alpha\beta}. \tag{B.51}$$

One can easily check that because of (2.15), (B.49) the second term vanishes. Equation (B.51) show us then that for the form

$$\begin{aligned} U &= e^G \star\Omega \Rightarrow \\ dU &= e^G(dG \wedge \star\Omega + d\star\Omega) \\ &= e^G(dG \wedge \star\Omega + \star\Omega \wedge dG) = 0. \end{aligned}$$

We know that

$$U = e^G L \wedge I$$

and since  $L$  is closed we have that

$$L \wedge d(e^G I) = 0. \tag{B.52}$$

Rescaling the vielbein in (B.24) as

$$e_\mu^a = e^{-\frac{1}{2}G - \frac{1}{2}H} \tilde{e}_\mu^a$$

we define the metric

$$h_{\mu\nu} = \delta_{ab} \tilde{e}_\mu^a \tilde{e}_\nu^b \tag{B.53}$$

and the two-form

$$\mathcal{J} = e^{G+H} I. \tag{B.54}$$

At this point we would like to fix our gauge so that

$$\begin{aligned} e^+ &= X(dt + C_m dx^m) + Bdy \\ e^- &= -X(dt + C_m dx^m) + Bdy \\ K &= -Xdt - XC \\ L &= \gamma dy \end{aligned}$$

and from equations (B.43) and (B.44) we draw the conclusion that

$$X = B^{-1} = h^{-2} = f_1^2 + f_2^2.$$

At this point the ten dimensional metric has the form

$$ds^2 = -\frac{1}{h^2}(dt + C)^2 + \gamma^2 h^2 dy^2 + e^{-G-H} h_{mn} dx^m dx^n - \frac{\gamma^\alpha}{\lambda} y e^G d\hat{\Omega}_3^2 - \frac{\gamma^\alpha}{\lambda} y e^{-G} d(\psi + A)^2, \quad m = 1, \dots, 4 \quad (\text{B.55})$$

where we used (B.36) to fix

$$e^H = -\frac{\gamma^\alpha}{\lambda} y.$$

We also know that the Euclidean space with metric

$$ds_4^2 = h_{mn} dx^m dx^n$$

is Kahler with complex structure defined by (B.54). In order to prove that  $\mathcal{J}$  is closed we split the exterior derivative as

$$d = \tilde{d} + d_y + d_t$$

where  $\tilde{d}$  only takes into account differentiation with respect to the coordinates  $x^m$ . Then from (B.52)

$$\begin{aligned} L \wedge d\mathcal{J} &= 0 \Rightarrow \\ \tilde{d}\mathcal{J} &= 0 \end{aligned} \quad (\text{B.56})$$

and from (B.42) we have

$$\mathcal{J}^m_p \mathcal{J}^p_n = -\delta_n^m$$

where in this equation we raise and lower indices using the metric (B.53).

From equation (B.46) we derive the constrain

$$f_1 V + f_2 Y = L \wedge K. \quad (\text{B.57})$$

And from equations (B.46), (B.47) and (B.48) we can derive the equations

$$\begin{aligned} i_L V &= f_1 K \\ i_K V &= -f_1 L \\ i_L Y &= f_2 K \\ i_K Y &= -f_2 L. \end{aligned}$$

We now use these relations and contract equations (B.37) and (B.38) with  $L^\kappa$  to derive

$$V = \frac{\lambda n}{\alpha} e^{\frac{1}{2}(H+G)} I + \frac{\lambda}{2\alpha} e^H e^{\frac{1}{2}(H-G)} \left( \mathcal{F} - \frac{1}{f_1^2 + f_2^2} i_L i_K \star_6 \mathcal{F} \right) + \frac{f_1}{f_1^2 + f_2^2} L \wedge K \quad (\text{B.58})$$

$$Y = \lambda e^{\frac{1}{2}(H-G)} I + \frac{f_2}{f_1^2 + f_2^2} L \wedge K.$$

In order to satisfy (B.57) we find that the following relations should be satisfied

$$\mathcal{F} - \frac{1}{f_1^2 + f_2^2} i_L i_K \star_6 \mathcal{F} = -2e^{G-H} \left( n + \frac{\kappa\alpha}{\lambda} \right) I. \quad (\text{B.59})$$

At this point we may use (B.58) and (B.59) to express

$$I = -\frac{1}{\kappa} e^{-\frac{1}{2}(H+G)} V + \frac{1}{\kappa} \frac{f_1 e^{-\frac{1}{2}(G+H)}}{f_1^2 + f_2^2} L \wedge K. \quad (\text{B.60})$$

One can calculate the derivative of  $V$  using equation (2.4) and (2.6),

$$dV_{\lambda\mu\nu} = e^{\frac{1}{2}(H-G)} L \wedge \mathcal{F} - \frac{1}{2} \frac{\kappa\alpha}{\lambda} e^{-\frac{1}{2}(H-G)} L \wedge I - \frac{1}{2} V \wedge dG \Rightarrow$$

$$d\left(e^{\frac{1}{2}G} V\right) = e^{\frac{1}{2}H} L \wedge \mathcal{F} - \frac{1}{2} \frac{\kappa\alpha}{\lambda} e^{-\frac{1}{2}(H-2G)} L \wedge I.$$

The above equation in combination with (B.60) and the fact that

$$dK = \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} dG \wedge K + dH \wedge K - (f_2^2 + f_1^2) dC$$

leads to the differential

$$d(e^{H+G} I) = e^H L \wedge \left( \frac{\lambda}{\kappa} dC - \frac{1}{\kappa} \mathcal{F} \right)$$

$$d\mathcal{J} = e^H L \wedge \left( \frac{\lambda}{\kappa} dC - \frac{1}{\kappa} \mathcal{F} \right)$$

We split the exterior derivative as

$$d = \tilde{d} + d_y + d_t$$

where  $\tilde{d}$  only takes into account differentiation with respect to the coordinates  $x^m$ . From the above we recover the fact that  $\mathcal{J}$  is closed and we find also an equation for the  $y$  dependence of this function

$$\partial_y \mathcal{J} = e^H \frac{\gamma}{\kappa} \left( \lambda \tilde{d}C - \mathcal{F} \right). \quad (\text{B.61})$$

**Eliminating the field strength.** We can use equation (B.18) to solve for the four form

$$4f_1 e^{-\frac{1}{2}(H-G)} \tilde{F}_{(4)} = d\Omega - e^{\frac{1}{2}(H-G)} Y \wedge \mathcal{F}$$

$$= -\frac{2f_1^2}{f_1^2 + f_2^2} dG \wedge K \wedge I + f_1 e^{-\frac{1}{2}(G+H)} L \wedge K \wedge dC - (f_2^2 + f_1^2) dC \wedge I$$

$$+ \lambda e^{H-G} I \wedge \mathcal{F} - e^{\frac{1}{2}(H-G)} \frac{f_1^2}{f_2} \frac{1}{f_1^2 + f_2^2} L \wedge K \wedge \mathcal{F}$$

From the Bianchi identity we have the constrain that  $W$  has to satisfy

$$\mathcal{F} \wedge d \left( e^{\frac{1}{2}(H-G)} Y \right) = 0. \quad (\text{B.62})$$

Contracting equation (B.10) with  $L^\mu$  we obtain the equation

$$L^\mu \partial_\mu f_1^2 = -\frac{1}{4} d\Omega_{\mu\rho\sigma\tau} L^\mu K^\rho I^{\sigma\tau}$$

$$\frac{1}{4} e^{\frac{1}{2}(G+H)} (f_1^2 + f_2^2) \mathcal{J}^{ij} \tilde{d}C_{ij} = \gamma f_1 \partial_y G + \gamma f_1 e^{-H} \partial_y e^H - e^{-\frac{1}{2}(H-G)} \frac{f_1}{f_2} \left( n + \frac{\kappa\alpha}{\lambda} \right)$$

where we used the constrain (B.59) and in the last line the raising of indices is done with the Kahler metric. Contracting equation (B.61) with  $\mathcal{J}$  we obtain

$$\mathcal{J}^{ij} \partial_y \mathcal{J}_{ij} = \frac{\gamma\lambda}{\kappa} e^H \mathcal{J}^{ij} dC_{ij} - \frac{\gamma}{\kappa} e^H \mathcal{J}^{ij} \mathcal{F}_{ij}$$

where the contraction is done using the rescaled metric (B.53). At this point we find it convenient to set

$$\begin{aligned} \kappa &= \lambda = 1 \\ \gamma^2 &= 1 \\ \gamma\alpha &= -1 \Rightarrow e^H = y. \end{aligned}$$

Continuing our analysis we have that

$$\partial_y \ln \det \mathcal{G} = 2 \frac{e^{-G}}{e^G + e^{-G}} \partial_y G + 2\gamma \frac{\gamma - n - \alpha}{y(1 + e^{2G})} + \frac{2\gamma}{y} (n + \alpha)$$

where

$$\mathcal{G} = \begin{vmatrix} \partial_z \partial_{\bar{z}} K & \partial_w \partial_{\bar{z}} K \\ \partial_z \partial_{\bar{w}} K & \partial_w \partial_{\bar{w}} K \end{vmatrix}$$

and  $K$  represents the Kahler potential for the four dimensional base manifold.

Defining the field

$$Z = \frac{1}{2} \tanh G$$

the previous equations may be written as

$$\partial_y \ln \det \mathcal{G} = \partial_y \ln \left( Z + \frac{1}{2} \right) + \frac{2\gamma}{y} \left( -Z + \frac{1}{2} \right) (\gamma - n - \alpha) + \frac{2\gamma}{y} (n + \alpha). \quad (\text{B.63})$$

We are now turning our attention to the component

$$\begin{aligned} F_{tx} &= \frac{e^{2G+H}}{4!} \epsilon_{tx}^{\pi\rho\sigma\tau} F_{\pi\rho\sigma\tau} \\ &= -\frac{1}{4} \frac{e^{\frac{3}{2}(G+H)}}{f_1} (f_1^2 + f_2^2)^2 \partial_y C_{x1} I_x^{x1}. \end{aligned}$$

Using equation (B.12) we have the constrain

$$I_x^{x^1} \partial_y C_{x^1} = -\frac{2f_1 f_2}{(f_1^2 + f_2^2)^2} \partial_x G$$

$$\partial_y C_{x^1} = \frac{1}{y} I_x^{x^1} \partial_{x^1} Z.$$

The consistency condition  $d^2 C = 0$  leads to the following expression between the scalar  $Z$  and the Kahler form  $\mathcal{J}$

$$y \partial_y \left( \frac{\partial_y \mathcal{J}}{y} \right) + \tilde{d} (\mathcal{J} \cdot \tilde{d} Z) = 0. \tag{B.64}$$

**The Ricci form of the Kahler base space.** We would like to construct the holomorphic 2-form  $\mathcal{B}$  for the 4 dimensional Kahler base space such that

$$\mathcal{B} \wedge \bar{\mathcal{B}} = \mathcal{J} \wedge \mathcal{J}.$$

For this reason we consider the bilinear

$$\mathcal{V}_{\mu\nu} = \varepsilon^T \gamma_{\mu\nu} \gamma_7 \varepsilon.$$

Our convention for the gamma matrices are such that

$$\gamma_\mu^* = \gamma_0 \gamma_\mu \gamma_0$$

$$\gamma_\mu^T = -\gamma_\mu.$$

Before we move on with our study we note the projection condition that the spinor has to satisfy as a result of the Fierz identities

$$\gamma_0 \gamma_7 \gamma_{\bar{y}} \varepsilon = \alpha \varepsilon. \tag{B.65}$$

Fierzing for  $\psi_4 = -\gamma_0 \gamma_{\kappa\lambda} \gamma_7 \gamma_0 \varepsilon^*$ ,  $\psi_2 = \varepsilon$ ,  $\psi_3 = \gamma_0 \varepsilon$  and  $\psi_1 = -\gamma_7 \gamma_{\mu\nu} \gamma_0 \varepsilon^*$  we find that

$$\mathcal{V} \wedge \bar{\mathcal{V}} = e^{-2(H+G)} (f_1^2 + f_2^2) \mathcal{J} \wedge \mathcal{J}.$$

We consider one more time the Fierz identity for  $\psi_4 = -\gamma_0 \gamma_{\mu\nu} \gamma_7 \gamma_0 \varepsilon^*$ ,  $\psi_2 = \varepsilon$ ,  $\psi_3 = \gamma_0 \varepsilon$  and  $\psi_1 = -\gamma_{\pi\rho\sigma} \varepsilon$  with which we may prove that

$$i_K \Omega \wedge \mathcal{V} = e^{-G-H} \mathcal{J} \wedge \mathcal{V} = 0$$

which proves that  $\mathcal{V}$  is holomorphic. At this point we note that we used (B.65) to show that

$$i_K \mathcal{V} = \sqrt{f_1^2 + f_2^2} \varepsilon^T \gamma_\nu \gamma_0 \gamma_7 \varepsilon = \alpha \sqrt{f_1^2 + f_2^2} \varepsilon^T \gamma_\nu \gamma_{\bar{y}} \varepsilon = 0$$

as a consequence of the antisymmetry of the antisymmetry of the gamma matrices. We may also use (A.4) to show that

$$\varepsilon^T \varepsilon = 0.$$

From the above analysis we see that

$$\mathcal{V} = e^{-(H+G)} \sqrt{f_1^2 + f_2^2} \mathcal{B}. \quad (\text{B.66})$$

We may now use (2.4) to show that

$$\begin{aligned} d\mathcal{V}_{\rho\mu\nu} &= inA \wedge \mathcal{V}_{\rho\mu\nu} - \frac{3}{2} e^{\frac{1}{2}(H-G)} \varepsilon^T \gamma_{[\mu\nu}{}^\lambda \varepsilon \mathcal{F}_{\lambda\rho]} - \iota \varepsilon^T (\gamma_{\rho\mu\nu} N - N \gamma_{\rho\mu\nu}) \gamma_7 \varepsilon \\ &= inA \wedge \mathcal{V}_{\rho\mu\nu} - \frac{3}{2} e^{\frac{1}{2}(H-G)} \varepsilon^T \gamma_{[\mu\nu}{}^\lambda \varepsilon \mathcal{F}_{\lambda\rho]} - \alpha e^{-\frac{1}{2}(H+G)} \varepsilon^T \gamma_{\rho\mu\nu} \gamma_7 \varepsilon - \frac{1}{2} \mathcal{V} \wedge d(H+G)_{\rho\mu\nu}. \end{aligned}$$

We are mostly interested in the case where the indices  $\rho, \mu, \nu$  belong to the four dimensional base space. For this reason one can write

$$\begin{aligned} d\mathcal{V}_{rnm} &= inA \wedge \mathcal{V}_{rnm} - \frac{3}{2} e^{\frac{1}{2}(H-G)} \sqrt{f_1^2 + f_2^2} C_{[m} \varepsilon^T \gamma_0 \gamma_n{}^l \varepsilon \mathcal{F}_{lr]} \\ &\quad - \alpha e^{-\frac{1}{2}(H+G)} \sqrt{f_1^2 + f_2^2} C_{[r} \varepsilon^T \gamma_0 \gamma_{mn]} \gamma_7 \varepsilon - \frac{1}{2} \mathcal{V} \wedge d(H+G)_{rnm}. \end{aligned} \quad (\text{B.67})$$

Where we used the fact that because of (B.65)

$$\bar{\varepsilon} \gamma_{\bar{r}\bar{m}\bar{n}} \varepsilon = 0.$$

There are three useful relations that we can derive from the projector (2.5)

$$\begin{aligned} -in e^{-\frac{1}{2}(H-G)} \mathcal{V}_{mn} - \frac{1}{2} e^{\frac{1}{2}(H-G)} \mathcal{V}_{[ma} \mathcal{F}_{n]}^a - \varepsilon^T \gamma_{mn}{}^l \varepsilon \partial_l H &= 0 \\ \alpha e^{-\frac{1}{2}(H+G)} \varepsilon^T \gamma_{mnl} \gamma_7 \varepsilon - in e^{-\frac{1}{2}(H-G)} \varepsilon^T \gamma_{mnl} \varepsilon - \frac{3}{2} e^{\frac{1}{2}(H-G)} \varepsilon^T \gamma_{[mn} \varepsilon \mathcal{F}_{l]}^a &= 0 \\ \alpha e^{-\frac{1}{2}(H+G)} \mathcal{V}_{mn} + \varepsilon^T \gamma_{mnl} \gamma_7 \varepsilon \partial^l H &= 0. \end{aligned}$$

At this points it is useful to note that one can easily prove that  $\mathcal{F}$  has to be a (1, 1) form and because  $\mathcal{V}$  is a holomorphic form we may write

$$\mathcal{V}_{[ma} \mathcal{F}_{n]}^a = \frac{1}{2} \iota I^{ab} \mathcal{F}_{ab} \mathcal{V}_{mn} = -2i e^{G-H} (n + \alpha) \mathcal{V}_{mn}.$$

From the first equation we have that

$$e^{-\frac{1}{2}(H+G)} \sqrt{f_1^2 + f_2^2} \varepsilon^T \gamma_0 \gamma_{mn} \gamma_7 \varepsilon = i\alpha \mathcal{V}_{mn}.$$

Using the second and third lines we may write

$$\frac{3}{2} e^{\frac{1}{2}(H-G)} \sqrt{f_1^2 + f_2^2} C_{[m} \varepsilon^T \gamma_0 \gamma_n{}^l \varepsilon \mathcal{F}_{lr]} = i(n + \alpha) C_{[m} \mathcal{V}_{mn]}.$$

Substituting the above two equations in (B.67) we obtain

$$\tilde{d}\mathcal{V} = i \left[ nA - (2\alpha + n)C - \frac{i}{2} \tilde{d}G \right] \wedge \mathcal{V}.$$

We may now use (B.66) to calculate

$$\tilde{d}\mathcal{B} = \left[ inA - i(2\alpha + n)C + \frac{1}{2} \tilde{d} \ln \left( Z + \frac{1}{2} \right) \right] \wedge \mathcal{B}.$$



From the above we have that the Ricci form of the Kahler manifold is given by

$$\begin{aligned} \mathcal{R} &= -n\mathcal{F} + (2\alpha + n) \tilde{d}C + \frac{1}{2}d \left( \mathcal{J} \cdot d \ln \left( Z + \frac{1}{2} \right) \right) \\ &= 2\alpha\mathcal{F} - \frac{\alpha}{y} (2\alpha + n) \partial_y \mathcal{J} + \frac{1}{2}d \left( \mathcal{J} \cdot d \ln \left( Z + \frac{1}{2} \right) \right) \end{aligned} \quad (\text{B.68})$$

which is compatible with equation (B.63).

We may now use the constrain (B.17) to write

$$\mathcal{F} = \tilde{d} \left( \mathcal{J} \cdot \tilde{d}\Phi \right).$$

Finally from (B.63) and (B.68) we have that

$$\begin{aligned} \ln \left| \frac{\partial_z \partial_{\bar{z}} K}{\partial_z \partial_{\bar{w}} K} \frac{\partial_w \partial_{\bar{z}} K}{\partial_w \partial_{\bar{w}} K} \right| &= \ln \left( -y \partial_y \left( \frac{\partial_y K}{y} \right) + y F'(y) \right) + \frac{2\alpha}{y} (2\alpha + n) \partial_y K - 2\Phi \\ &\quad - 2\alpha (n + \alpha) \ln y - 2\alpha (2\alpha + n) F(y) \end{aligned} \quad (\text{B.69})$$

where the scalars need to satisfy the two additional constraints

$$\begin{aligned} \partial \bar{\partial} \left[ Z + y \partial_y \left( \frac{\partial_y K}{y} \right) \right] &= 0 \\ \partial \bar{\partial} \Phi - \star_4 \partial \bar{\partial} \Phi &= -\frac{2}{y^2} (n + \alpha) \partial \bar{\partial} K \end{aligned}$$

and the function  $F(y)$  is such that

$$Z + \frac{1}{2} = -y \partial_y \left( \frac{\partial_y K}{y} \right) + y F'(y).$$

We also have to satisfy (2.2) and (B.62) which come from the Bianchi identity of the ten dimensional five form field. We can use the constrain (B.59) to show that

$$I \wedge \mathcal{F} = - (n + \alpha) e^{G-H} I \wedge I.$$

We can easily check that the constrains (2.2) and (B.62) are simultaneously solved when

$$\begin{aligned} e^{-2H} (n + \alpha) \partial_y \ln \left( Z + \frac{1}{2} \right) \mathcal{J} \wedge \mathcal{J} - e^{-H} (n + \alpha) \partial_y \left( \frac{1}{y^2} \mathcal{J} \wedge \mathcal{J} \right) & \quad (\text{B.70}) \\ -\alpha \mathcal{F} \wedge \mathcal{F} + \gamma \left( \frac{1}{2} - Z \right) \mathcal{F} \wedge \mathcal{F} &= 0. \end{aligned}$$

Comparing with equation (B.63) we see that we need to have

$$\mathcal{F} \wedge \mathcal{F} = 2 (n + 2\alpha) (n + \alpha) y^{-4} \mathcal{J} \wedge \mathcal{J}. \quad (\text{B.71})$$

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